Indian Statistical Institute, Bangalore
B. Math. First Year
Second Semester - Analysis II

Final Exam | Duration: 3 hours | Date : April 20, 2016

Max Marks: 50

(1) State and prove the Bolzano Weierstrass theorem for $\mathbb{R}^n$ (for $n > 1$). (2+4 = 6 marks)

(2) Let $n \geq 1$ be an integer. Consider the three metrics on $\mathbb{R}^n$, the l^1, l^2 and l^∞ metrics. Prove that the topologies on $\mathbb{R}^n$ induced by these three metrics are the same. (10 marks)

(3) True or False (give reasons):
  (a) $\mathbb{Q}$ (with the standard metric) is connected. (3 marks)
  (b) Any linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous. (3 marks)
  (c) Any real valued contnuous function on a compact metric space has a maximum and a minimum. (3 marks)
  (d) A continuous map from a compact metric space to any metric space is uniformly continuous. (3 marks)

(4) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \frac{xy}{\sqrt{x^2+y^2}}$ if $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Is f continuous at all points of $\mathbb{R}^2$? Is f differentiable at all points of $\mathbb{R}^2$? Does f have directional derivatives at $(0, 0)$ in every direction? Justify all your answers. (3+3+4=10 marks)

(5) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable map. Define $g : \mathbb{R}^n \to \mathbb{R}^{n+m}$ to be the map $g(x_1, \cdots, x_n) = (x_1, \cdots, x_n, f(x_1, \cdots, x_n))$. What is the derivative of g in terms of the derivative of f? Justify your answer. (6 marks)

(6) Let $E \subset \mathbb{R}^n$ be an open subset, and let $f : E \to \mathbb{R}$ be a real valued function such that all the partial derivatives of f are bounded in E. Prove that f is continuous in E. (6 marks)